MATH 20D Spring 2023 Lecture 27.

Solving Systems of Equations Using Eigenvectors II

Outline

1. Solving Systems of Equations Using Eigenvectors

2. Complex Eigenvalues

Contents

1 Solving Systems of Equations Using Eigenvectors

2 Complex Eigenvalues

- Suppose A is a 2×2 matrix with constant entries. In Lecture 24 we asked $\ldots$

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Leading Questions

(a) How do we write down a general solutions to the equation

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where $\mathbf{x}(t) = \mathrm{col}(x_1(t), x_2(t))$.

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Answers (assuming A has two linearly independent eigenvectors)

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Let $\mathbf{v}_1$, $\mathbf{v}_2$ be linearly independent eigenvectors of A with eigenvalues λ_1, λ_2.

(a) A general solution to (1) is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

(b) Let $X(t) = \begin{pmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{pmatrix}$ denote a fundamental matrix of (1) then

$$\mathrm{col}(C_1, C_2) = X(0)^{-1}\mathbf{x}_0.$$

Example

Solve the system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(t)$$

subject to the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} -2 \\ 1 + \frac{1}{2}\sqrt{5} \end{pmatrix}$$

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Solve the system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}(t)$$

subject to the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} -2/\sqrt{3} \\ 0 \end{pmatrix}$$

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2 Complex Eigenvalues

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$$ay''(t) + by'(t) + cy(t) = 0 \tag{2}$$
has linearly independent solutions
$$y_1(t) = e^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad y_2(t) = e^{\alpha t}\sin(\beta t)$$
where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.

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- We can rewrite (2) as the matrix equation
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and the coefficient matrix has eigenvalue $\alpha + i\beta$ and $\alpha - i\beta$.

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- We might expect that if A is a 2-by-2 matrix with complex conjugate eigenvalues, then the system

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

has two linearly indepedent solutions which can be expressed in terms of sines and cosines.

Theorem

Suppose A *is a* 2*-by-*2 *matrix with complex conjugate eigenvalues* $\lambda_1 = \alpha + i\beta$ *and* $\lambda_2 = \alpha - i\beta$.

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Suppose A is a 2-by-2 matrix with complex conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Let $\mathbf{v}$ be an eigenvector of λ_1 and write

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where $\mathbf{a}$ and $\mathbf{b}$ are vectors with real entries.

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define linearly independent solutions to the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$.

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Example

Solve the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}\mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Wrap up

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- Thank you for the quarter and good luck on all your finals!

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- Thank you for the quarter and good luck on all your finals!
- Please don't forget to fill out the CAPE survey.